

# ON THE METRIC STRUCTURE OF SOME NON-KÄHLER COMPLEX THREEFOLDS

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. We introduce a class of hermitian metrics with *Lee potential*, that generalize the notion of l.c.K. metrics with potential introduced in [14] and show that in the classical examples of Calabi and Eckmann of complex structures on  $S^{2p+1} \times S^{2q+1}$ , the corresponding hermitian metrics are of this type. These examples satisfy, actually, a stronger differential condition, that we call *generalized Calabi-Eckmann*, condition that is satisfied also by the *Vaisman* metrics (previously also referred to as *generalized Hopf manifolds*). This condition means that, in addition to being with Lee potential, the torsion of the *characteristic* (or Bismut) connection is parallel. We give a local geometric characterization of these generalized Calabi-Eckmann metrics, and, in the case of a compact threefold, we give detailed informations about their global structure. More precisely, the cases which can not be reduced to Vaisman structures can be obtained by deformation of locally homogenous hermitian manifolds that can be described explicitly.

## 1. INTRODUCTION

The non-Kählerianity of a hermitian metric is measured by the exterior differential of the Kähler form, which decomposes as

$$d\omega = -2\theta \wedge \omega + \Omega_0, \quad (1)$$

where  $\theta \in \Lambda^1 M$  is the *Lee form* of  $\omega$ , and  $\Omega_0$  the *trace-free* part of  $d\omega$ . In complex dimensions greater than 2, the vanishing of  $\Omega_0$  implies that  $\theta$  is closed or, equivalently, that the hermitian metric is *locally conformally Kähler* (l.c.K.), and this kind of metrics have been studied for a long time by many specialists [16], [9], [4], [14], to name just a few.

In fact, the first examples of non-Kähler compact complex manifolds were given by the Hopf manifolds  $\mathbb{C}^n \setminus \{0\}$  divided by a nontrivial linear diagonal contraction, which are of this hermitian type. Moreover, the standard flat metric on  $\mathbb{C}^n$  induces, by renormalization, a *Vaisman* (formerly: *generalized Hopf*) metric on the quotient, i.e. it is not only l.c.K., but its Lee form is not just closed, but parallel as well. This lead to the former name of the structure, name which was proven inappropriate in [4], where examples of Hopf surfaces not admitting such a Vaisman metric were given. Also in [4], the stability to small deformations of the class of Vaisman, or even l.c.K. manifolds was disproven.

However, in [14], Ornea and Verbitsky introduced the concept of *l.c.K. metrics with potential*, or l.c.K.p., which form a subclass of l.c.K. metrics containing the Vaisman structures, subclass which turns out to be stable to deformations [14]. This class of metrics can be characterized by the fact that its Kähler form is determined by the 1-form  $\theta$  through the following relation:

$$\omega = c \left( \theta \wedge J\theta + \frac{1}{2} d(J\theta) \right), \quad \text{and } d\theta = 0, \quad (2)$$

where  $\theta$  is the (closed) Lee form and  $c > 0$  is a constant, hence  $\omega$  is l.c.K. Here, and throughout the paper,  $\omega := g(J\cdot, \cdot)$  is the Kähler form of the metric  $g$ ,  $J$  is the complex structure, and  $J\theta := -\theta \circ J$ .

If we consider the exterior derivative of (2), and we no longer assume that the (real) Lee form  $\theta$  is closed, but only that  $\bar{\partial}\theta^{(0,1)} = 0$  (equivalent to the fact that  $d\theta$  or  $d(J\theta)$  is a  $(1,1)$ -form), then we obtain

$$d\omega = c(-\theta \wedge d(J\theta) + d\theta \wedge (J\theta)) \quad \text{and} \quad \bar{\partial}\theta^{(0,1)} = 0, \quad (3)$$

and we call the corresponding metric a *metric with Lee potential* (in short LP). Note that, if (3) is satisfied for  $\theta$ , then a similar relation holds (with some constant  $c'$ ) for any non-zero linear combination of  $\theta$  and  $J\theta$ . In (3), therefore, the Lee form does not play a privileged role, as in (2), but only the “complex line” determined by  $\theta$  occurs as the set of potentials for  $d\omega$ .

Note that from a topological point of view, the *non-Kählerianity* of an l.c.K. metric (or, more particularly, l.c.K.p. or even Vaisman) lies in the first cohomology group of the manifold. More precisely, any such metric on a simply-connected manifold is automatically Kähler.

On the other hand, in 1953, Calabi and Eckmann gave an example of compact, non-Kähler complex homogeneous structure on a product of odd-dimensional spheres, [8]. This generalizes the classical example of Hopf manifolds, in which case one of the factors is a circle, but, if the dimension of both factors is at least 3, it produces compact, simply-connected examples of non-Kähler manifolds.

The first remark that we make in this paper is that the Calabi-Eckmann manifolds, with their standard product metric, satisfy the equations (3), and are, hence, metrics with Lee potential. On the other hand, they generalize the Vaisman structures in the following way:

**Definition 1.1.** *An LP hermitian metric that has a non-zero Lee form and such that the torsion of its characteristic (or Bismut) connection is parallel is called generalized Calabi-Eckmann (or GCE).*

Recall that the characteristic connection of a hermitian metric is unique with the property that it preserves the hermitian structure and that its torsion tensor is totally skew-symmetric (a 3-form  $T$ , of type  $(2,1) + (1,2)$ ). By contraction with the Kähler form we obtain  $J$  times the Lee form which is, in our GCE case, *parallel with respect to the characteristic connection*. Only if the Lee form is closed is this form parallel also with respect to the Levi-Civita connection (hence the Vaisman manifolds form a subclass of GCE manifolds).

We obtain thus that the GCE manifolds form a subclass of LP manifolds, but, in the case of complex threefolds, they can be deformed to become either Vaisman manifolds, local products of a Vaisman surface with a Riemann surface, or local products of *Sasakian* manifolds, see section 3).

The latter (generic) case will be described in detail; we show, in particular, that such a local Sasakian product is always a deformation of a locally homogeneous one, and the underlying manifolds are 3-dimensional analogues to the non-Kähler elliptic surfaces and Hopf surfaces, Theorem 4.5.

Because the GCE structures are related to Sasakian geometry, as much as the Vaisman structures are, it is not surprising that the results in this article are somewhat similar to the ones in [4]. However, the details are only indirectly related, and – more important –, we do not have a classification of compact complex threefolds to use, as we did for surfaces [4]. Therefore, although the fundamental group of a GCE threefold is not fully understood, most topological and hermitian properties of these complex manifolds can be described in detail.

The paper is organized as follows: In the first section we recall some facts about hermitian geometry and the characteristic connection. In the second, we study the case of parallel characteristic torsion and obtain a local geometric characterization of GCE manifolds in all dimensions, extending the results of [2], [15]. In the third, we recall elements of Sasakian geometry and facts about l.c.K.p. and Vaisman geometry. Finally, we prove Theorem 4.5 by examining the various cases that can occur on a compact GCE threefold.

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## 2. HERMITIAN GEOMETRY: BASICS AND THE CHARACTERISTIC CONNECTION

Let  $(M, J)$  be a complex manifold, i.e.  $J$  is an endomorphism of  $TM$  of square  $-I$  and *integrable*, i.e. its Nijenhuis tensor  $N^J \in \Lambda^2 M \otimes TM$ , defined by

$$4N^J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \quad (4)$$

vanishes identically. Let  $g$  be a hermitian metric, i.e.  $g$  is a (positive definite) Riemannian metric, such that  $g(J\cdot, J\cdot) = g$ . In fact, a hermitian metric is a reduction to  $U(m)$  (if  $\dim_{\mathbb{C}} M = m$ ) of the structure group of  $TM$ . Let  $\omega := g(J\cdot, \cdot)$  be the Kähler form of the hermitian metric  $g$ . It is a  $(1, 1)$ -form, and the space of  $k$ -forms on  $M$  decomposes in the following  $U(m)$ -irreducible components:

$$\Lambda^k M \otimes \mathbb{C} = \bigoplus_{p=0}^k \Lambda^{p, k-p} M, \text{ and } \Lambda^{p, k-p} M = \Lambda_0^{p, k-p} M \oplus \Lambda^{p-1, k-p-1} M, \quad \forall p = 1, \dots, k-1,$$

where  $\Lambda_0^{p, q} M$ , for  $p, q > 0$ , is the kernel of the contraction with  $\omega$ , i.e. of the map from  $\Lambda^{p, q} M$  to  $\Lambda^{p-1, q-1} M$  defined by

$$\alpha \longmapsto \sum_{i=1}^n \alpha(e_i, J e_i, \dots),$$

where  $\{e_i\}$  is a hermitian-orthogonal basis of  $TM$ .

Note that the real part of a  $(p, q)$ -form is equally the real part of its conjugate, which is a  $(q, p)$ -form, hence we denote by  $\Lambda^{(p, q) + (q, p)} M$  the real part of the sum of the vector spaces  $\Lambda^{p, q} M$  and  $\Lambda^{q, p} M$ . For  $p = q$ , our convention is to use *real* forms, so we denote by  $\Lambda^{p, p} M$  the space of *real*  $(p, p)$ -forms.

The operator  $J \in \text{End}(TM)$  can be considered at the same time as an element in the Lie group  $U(n)$  and in its Lie algebra  $\mathfrak{u}(n)$ . We denote by  $J\alpha$  the Lie algebra action on  $J$  on the tensor  $\alpha$ , and by  $\mathcal{J}\alpha$  the Lie group action of  $J$  on the same tensor. If  $\alpha$  is a  $k$ -form, then

$$\begin{aligned} (\mathcal{J}\alpha)(X_1, \dots, X_k) &= (-1)^k \alpha(JX_1, \dots, JX_k), \\ (J\alpha)(X_1, \dots, X_k) &= - \sum_{i=1}^k \alpha(X_1, \dots, JX_i, \dots, X_k). \end{aligned}$$

We have then

$$\begin{aligned} \alpha \in \Lambda^1 M &\Rightarrow \mathcal{J}\alpha = J\alpha \\ \alpha \in \Lambda^{1, 1} M &\Leftrightarrow \mathcal{J}\alpha = \alpha \Leftrightarrow J\alpha = 0 \\ \alpha \in \Lambda^{2, 0+0, 2} M &\Leftrightarrow \mathcal{J}\alpha = -\alpha \Leftrightarrow J\alpha \in \Lambda^{2, 0+0, 2} M \\ \alpha \in \Lambda^{2, 1+1, 2} M &\Leftrightarrow \mathcal{J}\alpha = J\alpha \end{aligned} \quad (5)$$

We are next interested in the Hodge  $*$  operator and its relations to  $J$ . Recall:

**Definition 2.1.** Let  $(M, g)$  be an oriented Riemannian manifold and let  $v_g \in \Lambda^n M$  be its canonical volume form and denote by  $\langle \cdot, \cdot \rangle$  the induced scalar product on any tensor space of  $M$ . The Hoge star operator  $* = *_g : \Lambda^p M \rightarrow \Lambda^{n-p} M$  is chracterized by the property

$$\forall \beta \in \Lambda^{n-p} M, \langle *\alpha, \beta \rangle v_g = \alpha \wedge \beta.$$

Equivalently, if  $e_1, \dots, e_n$  is a  $g$ -orthonormal basis of  $TM$ ,

$$*(e_1 \wedge \dots \wedge e_p) = e_{p+1} \wedge \dots \wedge e_n. \quad (6)$$

On an almost hermitian manifold, we can choose the basis to be adapted to the complex structure, i.e.,  $Je_{2i-i} = e_{2i}$ ,  $\forall i = 1, \dots, m$ , and we have

**Proposition 2.2.** On an almost hermitian manifold with Kähler form  $\omega$ , we have the following identities:

$$v_g = \frac{\omega^m}{m!}, \quad (7)$$

$$*\alpha = J.\alpha \wedge \frac{\omega^{m-1}}{(m-1)!} \quad \forall \alpha \in \Lambda^1 M, \quad (8)$$

$$*\frac{\omega^k}{k!} = \frac{\omega^{m-k}}{(m-k)!}, \quad (9)$$

$$*\alpha = -\alpha \wedge \frac{\omega^{m-2}}{(m-2)!}, \quad \forall \alpha \in \Lambda_0^{1,1} M, \quad (10)$$

$$*\alpha = \alpha \wedge \frac{\omega^{m-2}}{(m-2)!}, \quad \forall \alpha \in \Lambda^{2,0+0,2} M, \quad (11)$$

$$*(\alpha \wedge \omega) = J.\alpha \wedge \frac{\omega^{m-2}}{(m-2)!} \quad \forall \alpha \in \Lambda^1 M, \quad (12)$$

$$*\alpha = \mathcal{J}\alpha \wedge \frac{\omega^{m-3}}{(m-3)!} \quad \forall \alpha \in \Lambda_0^{2,1+1,2} M. \quad (13)$$

*Proof.* We choose a hermitian basis  $e_1, Je_1, \dots, e_m, Je_m$  of  $TM$  and use (6) to compute the image through  $*$  of some generators of the corresponding form spaces; indeed, we denote  $e_i \wedge Je_i$  by  $\omega_i$  and have:

$$\omega = \sum_{i=1}^m \omega_i, \quad \omega^k = \sum_{1 \leq i_1 \dots i_k \leq m} \frac{m!}{(m-k)!} \omega_{i_1} \wedge \dots \wedge \omega_{i_k}.$$

On the other hand, the spaces of forms considered in the Proposition have the following sets of generators:

$$\{e_i, Je_i\}_i \text{ generate } \Lambda^1 M, \quad (14)$$

$$\{\omega_i - \omega_j, e_i \wedge e_j + Je_i \wedge Je_j\}_{i \neq j} \text{ generate } \Lambda_0^{1,1} M, \quad (15)$$

$$\{e_i \wedge e_j - Je_i \wedge Je_j\}_{i \neq j} \text{ generate } \Lambda^{2,0+0,2} M, \quad (16)$$

and use that the map

$$\Lambda^1 M \otimes \Lambda^{1,1} M \rightarrow \Lambda^{2,1+1,2} M, \quad \theta \otimes \alpha \mapsto \theta \wedge \alpha$$

is surjective, in order to get that

$\{e_i \wedge (\omega_j - \omega_k), Je_i \wedge (\omega_j - \omega_k), e_i \wedge e_j \wedge e_k + e_i \wedge Je_j \wedge Je_k, Je_i \wedge e_j \wedge e_k + Je_i \wedge Je_j \wedge Je_k\}$ ,  
with  $i, j, k \in \{1, \dots, m\}$  distinct, generate  $\Lambda_0^{2,1+1,2} M$ .

The claimed identities follow by straightforward computations.  $\square$

The *Lee form*  $\theta$  of  $\omega$  is the unique 1-form that satisfies one of the following equivalent equations:

$$d\omega = -2\theta \wedge \omega + \Omega_0, \text{ with } \Omega_0 \in \Lambda_0^{(2,1)+(1,2)}, \iff \theta = -\frac{1}{2(m-1)}J\delta^g\omega. \quad (17)$$

(The normalization factor  $-2$  in the first equation is consistent with our convention for the Sasakian structures – see below.) Here, and below, the *codifferential* is defined, on the even-dimensional manifold  $M$ , by the usual formula  $\delta := - * d *$ .

The following result is classical:

**Proposition 2.3.** *In the decomposition above, if  $\dim_{\mathbb{R}} M = 4$ , then  $\Omega_0 = 0$ ; if  $\dim_{\mathbb{R}} M \geq 6$  and  $\Omega_0 = 0$ , then  $\theta$  is closed and the metric is l.c.K. (i.e., for each point  $x \in M$ , there exists a Kähler metric  $g_x$ , defined on an open set  $U_x$  containing  $x$ , which is conformally equivalent to  $g$ :  $g_x = e^f g$ , for some function  $f : U_x \rightarrow \mathbb{R}$ .)*

The proof is based on the fact that the wedge product with  $\omega$ ,

$$\omega \wedge \cdot : \Lambda^{(p,q)} M \rightarrow \Lambda^{(p+1,q+1)}$$

is injective iff  $p, q < \dim_{\mathbb{C}} M$ .

A theorem by Gauduchon [10] implies that, on a compact l.c.K. manifold, there is a unique metric (the *standard*, or the *Gauduchon* metric), conformally equivalent with the original one, for which the Lee form is harmonic.

A special class of Gauduchon metrics consists of the *Vaisman structures*, for which the Lee form is non-zero and parallel. Formerly called *generalized Hopf manifolds*, the corresponding complex manifolds admit a non-vanishing holomorphic vector field (whose real part is the metric dual to the Lee form), which is a strong topological condition implying the vanishing of the Euler characteristic  $\chi(M)$ . Another necessary condition for the existence of a non-globally conformally Kähler l.c.K. metric is a non-zero first de Rham cohomology vector space, since adding  $df$  to the Lee form corresponds to multiplying the metric with  $e^{-2f}$ .

In fact, for  $M$  a compact complex surface, the first Betti number is even iff the surface admits a Kähler metric. The author classified the l.c.K. complex surfaces with  $\chi(M) = 0$ , and also the Vaisman structures on them (when they exist) in [4]. A conclusion of this series of results is that neither the class of Vaisman surfaces, nor the larger class of l.c.K. surfaces is stable by small deformations. In [14], Ornea and Verbitsky introduced an intermediate class, of *l.c.k. metrics with potential*, for which the Kähler form is determined by the (closed) Lee form:

$$\omega = c \left( \theta \wedge J\theta + \frac{1}{2}d(J\theta) \right), \quad c > 0 \text{ constant and } d\theta = 0.$$

**Definition 2.4.** *A hermitian metric with Kähler form  $\omega$  and Lee form  $\theta$  is a metric with Lee potential (LP) iff*

$$d\omega = c(d\theta \wedge J\theta - \theta \wedge d(J\theta)), \quad c > 0 \text{ constant, and } \bar{\partial}\theta^{(0,1)} = 0,$$

where  $\theta^{(0,1)}$  is the  $(0,1)$ -part of the real form  $\theta$ :

$$\theta^{(0,1)} := \frac{1}{2}(\theta + iJ\theta).$$

Note that the condition  $\bar{\partial}\theta^{(0,1)} = 0$  is equivalent to  $d\theta$  (and also  $d(J\theta)$ ) being of type  $(1,1)$ .

The LP metrics correspond to a special form of a refined version of the decomposition (17). Note that, although  $\Omega_0 = 0$  in (17) implies that  $d\theta = 0$ , the converse is not true. On the other hand, the  $\bar{\partial}$ -closure of the  $(0,1)$ -form  $\theta^{(0,1)}$  is a more general fact than the closure of some linear combination of  $\theta$  and  $J\theta$ , and the exactness of  $\theta^{(0,1)}$  is equivalent to the vanishing of the Dolbeault class  $[\theta^{(0,1)}] \in H^{(0,1)}M$ .

Before showing that the Calabi-Eckmann complex structures on  $S^{2p+1} \times S^{2q+1}$ ,  $p, q > 0$  admit a LP metric (the standard product metric, or some straightforward modification of it), we need to recall some basic facts of *Sasakian geometry*.

**Definition 2.5.** A manifold  $(N^{2n+1}, g, \xi)$  is Sasakian iff  $g$  is a Riemannian metric,  $\xi$  is a unit Killing vector field (the Reeb field) whose covariant derivative is a complex structure on  $H := \xi^\perp$ , which is integrable in the CR sense.

Recall that a CR manifold is an odd-dimensional manifold  $N^{2m+1}$  endowed with a distribution of hyperplanes  $H$  which is a *contact structure* (i.e.,  $\eta \wedge (d\eta)^n \neq 0$  for any 1-form  $\eta$  whose kernel is  $H$  –  $\eta$  is then called a *contact form*), and  $J : H \rightarrow H$  is a complex structure on  $H$  for which  $d\eta|_H$  is of type  $(1,1)$  for every contact form  $\eta$ . The CR structure is *integrable* iff the Nijenhuis tensor  $N^J \in \text{Hom}(\Lambda^2 H, H)$  of  $J$ , defined by (4), vanishes identically.

It is well-known that the round metric of an odd-dimensional sphere  $S^{2n+1}$  is Sasakian, and the corresponding Reeb vector field generates the circle action on  $S^{2n+1}$  defined by the of multiplication of an element of  $S^{2n+1} \subset \mathbb{C}^{n+1}$  with a complex number of norm 1, and the basic Calabi-Eckmann structure on  $S^{2p+1} \times S^{2q+1}$  is very simple to describe in hermitian terms:

**Proposition 2.6.** Let  $(N_1, g_1, \xi_1)$  and  $(N_2, g_2, \xi_2)$  be two Sasakian manifolds of dimensions  $2n_1 + 1$ , resp.  $2n_2 + 1$ , and let  $(H_i, J_i)$  be their CR structures. Then the product metric  $g := g_1 + g_2$  is hermitian with respect to the following almost complex structure, which is actually integrable:

$$JX := J_i X, \quad X \in H_i; \quad (18)$$

$$J\xi_1 := \xi_2. \quad (19)$$

The proof is straightforward.

Recall that the product of a Sasakian manifold with a circle is Vaisman, and we see that this is the particular case of the above Proposition, where one of the Sasakian manifolds is 1-dimensional. All *Sasakian automorphisms* (i.e., the isometries that preserve the Reeb field) of the factors become thus hermitian isometries of  $(M, J, g)$ , in particular the standard Calabi-Eckmann complex manifold  $S^{2p+1} \times S^{2q+1}$  is homogeneous, and its hermitian automorphism group is  $U(p+1) \times U(q+1)$ .

In fact, Calabi and Eckmann gave a family of complex structures on  $S^{2p+1} \times S^{2q+1}$ , depending on one parameter  $\alpha \in \mathbf{H} := \{\alpha \in \mathbb{C} \mid \Im \alpha > 0\}$ , by setting in the Proposition 2.6

$$J_\alpha \xi_1 := \text{Re}(\alpha)\xi_1 + \text{Im}(\alpha)\xi_2$$

instead of  $J\xi_1 = \xi_2$ , and extending it by linearity and such that  $J_\alpha^2 = -I$ . For these complex structures (also integrable), the product metric is not hermitian any more, but a straightforward modification

$$\begin{aligned} g_\alpha(\xi_i, X) &:= 0, & X &\in H_1 \oplus H_2 \\ g_\alpha(X, Y) &:= g(X, Y), & X, Y &\in H_1 \oplus H_2 \\ g_\alpha(\xi_1, \xi_1) &:= g(J_\alpha \xi_1, J_\alpha \xi_1) &:= 1 \\ g_\alpha(\xi_1, J_\alpha \xi_1) &:= 0 \end{aligned} \tag{20}$$

is, and the hermitian automorphism group doesn't change.

These more general complex structures, and their associated hermitian metrics, will be locally characterized in the next section.

### 3. PARALLEL CHARACTERISTIC TORSION. GENERALIZED CALABI-ECKMANN STRUCTURES

We intend to (locally) characterize the Calabi-Eckmann complex and hermitian structures by a differential-geometric condition on the hermitian manifold  $(M, J, g)$ . Recall that, if  $(M, J, g)$  is Vaisman, then its Lee form, being parallel, generates a local product structure (by the decomposition Theorem of de Rham) which is easily checked to be locally isomorphic to the construction of Proposition 2.6 (for  $n_2 = 0$ ).

For  $m_1, m_2 > 0$ , we do not get any parallel 1-form and, if  $\alpha \in \mathbf{H}$  is generic, there is no parallel distribution in the Calabi-Eckmann manifold  $(S^{2p+1} \times S^{2q+1}, J_\alpha, g_\alpha)$  for the *Levi-Civita connection*.

The idea is then to use another canonical connection on this manifold, one that respects both the metric and the complex structure: the *characteristic* connection. This notion arises in a more general context than the one of hermitian geometry:

**Definition 3.1.** *Let  $M$  be a  $n$ -dimensional manifold with a pseudo-Riemannian  $G$ -structure on it, i.e. a reduction to  $G$  of the structure group of the orthogonal frame bundle of some pseudo-Riemannian metric  $g$ , for a given representation  $\rho : G \rightarrow O(\mathbb{R}^n, g_0)$ . A connection  $\nabla$  on  $M$  is called characteristic iff its torsion  $T^\nabla \in \Lambda^2 M \otimes TM \xrightarrow{g} \Lambda^2 M \otimes \Lambda^1 M$  is totally skew-symmetric, i.e.,  $T^\nabla \in \Lambda^3 M$ .*

Note that, in order to identify  $TM$  with its dual,  $G$  needs to preserve some non-degenerate bilinear form  $g_0$  on  $\mathbb{R}^n$ , and the Lie algebra of  $G$  consists – under this identification – of skew-symmetric bilinear forms iff  $g_0$  is symmetric. Therefore the restriction to pseudo-Riemannian  $G$ -structures is necessary.

Because the map

$$\nabla \mapsto T^\nabla,$$

associating to a pseudo-Riemannian connection its torsion is injective, the set of characteristic connections, if non-empty, is an affine space modelled on the space of sections of

$$\mathbb{T} := T^*M \otimes \mathfrak{g}(M) \cap \Lambda^2 M \otimes T^*M \subset \Lambda^3 M,$$

where  $\mathfrak{g}(M) \subset \Lambda^2 M$  is (isomorphic to) the adjoint bundle of the  $G$ -structure. Note that the first term on the left hand side is skew-symmetric in the last 2 arguments and the second term in the first two. For example, if  $\mathfrak{g} = \mathfrak{so}(p, q)$ , then  $\mathbb{T} = \Lambda^3 M$ , and if  $\mathfrak{g} = \mathfrak{u}(p, q)$ , then  $\mathbb{T} = 0$ .

Therefore, if an almost pseudo-hermitian manifold admits a characteristic connection, then it is unique. The torsion of this connection is then the  $(2, 1) + (1, 2)$ -form  $-\mathcal{J}d\omega = -J.d\omega$



(see the conventions in the previous section). Indeed, the differential of the  $\nabla$ -parallel form  $\omega$  can be computed in term of the torsion

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

of  $\nabla$  by the formula

$$d\omega(X, Y, Z) = \omega(T(X, Y), Z) + \omega(T(Y, Z), X) + \omega(T(Z, X), Y). \quad (21)$$

But  $\omega(T(X, Y), Z) = -g(T(X, Y), JZ) = -T(X, Y, JZ)$ , and  $T$  is skew-symmetric, hence

$$d\omega = J.T = \mathcal{J}T \iff T = -\mathcal{J}d\omega.$$

Coming back to the Calabi-Eckmann construction, or, more generally, to the one in Proposition 2.6 (possibly with the hermitian structure  $(J_\alpha, g_\alpha)$ ), we will show that the characteristic torsion is parallel:

**Proposition 3.2.** *In the Sasakian product of Proposition 2.6, and also for the modified hermitian structure (20), the hermitian structure has parallel characteristic torsion.*

*Proof.* Let us first remark that, if a  $G$ -structure has a unique characteristic connection  $\nabla$  with parallel torsion  $T$ , any  $\nabla$ -parallel 1-form  $\eta$  is closed iff it is parallel for the Levi-Civita connection, because  $d\eta$  is a multiple of  $T(\eta^\sharp)$ , where  $\eta^\sharp$  is the dual vector to  $\eta$ .

Then, let us consider a Sasakian manifold  $(N, g, \xi)$ . Because the Vaisman manifold  $N \times S^1$  has parallel characteristic torsion  $J\theta \wedge \omega$ , then the Lee form is parallel w.r.t. both the Levi-Civita connection and the characteristic connection  $\nabla$ . Because  $\theta$  (which is a non-zero multiple of the length form on the  $S^1$  factor) is  $\nabla$ -parallel, then  $\nabla$  induces a characteristic connection w.r.t. the Sasakian structure on the factor  $N$  as well. Conversely, the product of the characteristic connections of two factors is a characteristic connection for the product  $G$ -structure.

From here we infer that the characteristic connection of a Sasakian manifold is unique, that its torsion is parallel, and that the product of two Sasakian manifolds has parallel characteristic torsion as well.

The fact that the modifications  $(J_\alpha, g_\alpha)$  of the basic Sasakian product of Proposition (2.6) are LP and have parallel characteristic torsion will be a consequence of Proposition 3.6 below.  $\square$

We define thus:

**Definition 3.3.** *A hermitian manifold  $(M, g, J)$  with non-zero Lee potential and with parallel characteristic connection is called generalized Calabi-Eckmann, in short GCE.*

Let  $(M, g, J, \omega)$  be a hermitian manifold with characteristic connection  $\nabla$ , and suppose that its torsion  $T = -\mathcal{J}d\omega$  is a  $\nabla$ -parallel  $(2, 1) + (1, 2)$ -form. Later, we will focus on the GCE case, but for now, we just assume that the Lee form  $\theta$  of  $\omega$  is non-zero. Using (17), we obtain the following decomposition for  $T$ :

$$T = -2J\theta \wedge \omega - \mathcal{J}\Omega_0. \quad (22)$$

Therefore, the Lee form  $\theta$  is parallel (and not zero). To keep the notation simple, and also because in a LP manifold the Lee form itself is not so relevant as the complex line generated by it, we will write

$$T = \eta \wedge \omega - \mathcal{J}\Omega_0,$$



with  $\eta := -2J\theta$ , and recall that the Lee form of  $\omega$  can be retrieved as  $\frac{1}{2}J\eta$ . We will also denote by  $\eta$  the dual vector field, and, by our convention in Section 2,  $J\eta$  (as a vector field) coincides with the 1-form denoted by  $J\eta$ .

In order to study the algebraic structure of  $T$ , we will use two facts about a characteristic connection with parallel torsion: the first one is the (algebraic) Bianchi identity, which follows from

$$d^\nabla I = T,$$

where  $I$  is the identity of  $TM$ , seen as a 1-form with values in  $TM$ , and  $d^\nabla$  is the exterior differential of a  $k$ -form with values in  $TM$ :

$$\begin{aligned} d^\nabla \alpha(X_0, \dots, X_k) &:= \sum_{i=0}^k (-1)^i \nabla_{X_i} \left( \alpha(X_0, \dots, \hat{X}_i, \dots, X_k) \right) + \\ &+ \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], \dots, \hat{X}_i, \dots, \hat{X}_j, \dots). \end{aligned} \quad (23)$$

It is well known that

$$d^\nabla d^\nabla \alpha = R^\nabla \wedge \alpha,$$

where  $R^\nabla \wedge \alpha$  is the exterior product of the 2-form  $R^\nabla$  (with values in  $\text{End}(TM)$ ) with the  $k$ -form  $\alpha$ .

Therefore

$$R^\nabla \wedge I = d^\nabla d^\nabla I = d^\nabla T,$$

more precisely

$$R_{X,Y}^\nabla Z + R_{Y,Z}^\nabla X + R_{Z,X}^\nabla Y = T(T(X,Y), Z) + T(T(Y,Z), X) + T(T(Z,X), Y).$$

The left hand side is the Bianchi expression in the curvature  $R^\nabla$ , and the right hand side is – in our case, where  $T$  is totally skew-symmetric – a 4-form  $\Omega$ , defined as

$$\Omega := \frac{1}{2} \sum_{i=1}^{2m} T(\varepsilon_i, \cdot, \cdot) \wedge T(\varepsilon_i, \cdot, \cdot), \quad (24)$$

where  $\{\varepsilon_i \mid 1 \leq i \leq 2n\}$  is an orthonormal basis of  $TM$ . Thus

$$g(R_{X,Y}^\nabla Z + R_{Y,Z}^\nabla X + R_{Z,X}^\nabla Y, V) = \Omega(X, Y, Z, V). \quad (25)$$

Note that, if we compute the exterior differential of the 3-form  $T$  using a formula similar to (21), we obtain

$$dT = 2\Omega.$$

The second fact that we will use to decompose  $T$  involves the  $\nabla$ -codifferential of a  $k$ -form:

$$\delta^\nabla \alpha := - \sum_{i=1}^{2n} \varepsilon_i \lrcorner \nabla_{\varepsilon_i} \alpha.$$

This codifferential is obviously zero on a  $\nabla$ -parallel form (fact which does not hold for  $d^\nabla$  defined above). The  $\nabla$ -codifferential is related to the usual  $g$ -codifferential (defined as  $\delta := - * d *$  on an even-dimensional manifold), as shown in [1]:

**Proposition 3.4.** [1] *Let  $(M, g)$  be a Riemannian manifold and  $\nabla$  a metric connection with skew-symmetric torsion  $T$ . Then*

$$\delta^\nabla \alpha - \delta \alpha = \frac{1}{2} \sum_{i < j} (\varepsilon_i \lrcorner \varepsilon_j \lrcorner \alpha) \wedge (\varepsilon_i \lrcorner \varepsilon_j \lrcorner T),$$

for any  $k$ -form  $\alpha$ ,  $k > 1$ . In particular  $\delta^\nabla T = \delta T = 0$ .

Assuming  $(M, g, J)$  is a hermitian manifold with characteristic connection  $\nabla$ , we can now state the following

**Proposition 3.5.** *Let  $(M, g, J)$  be a  $2n$ -dimensional hermitian manifold such that the torsion  $T$  of the characteristic connection  $\nabla$  is  $\nabla$ -parallel. Assume, moreover, that the Lee form  $\theta := \frac{1}{2}J\eta$  of  $\omega$  is not zero. Then*

$$T = \eta \wedge \omega_+ + \mathcal{J}\eta \wedge \omega_- + T_0, \quad (26)$$

where

- (1)  $\eta$  is a  $\nabla$ -parallel form dual to a Killing vector field of constant length, which is moreover real-holomorphic (therefore  $[\eta, J\eta] = 0$ ); denote by  $E : \mathbb{R}\eta \oplus \mathbb{R}J\eta$  be the complex (integrable) distribution spanned by the Lee vector field and let  $H$  be its orthogonal complement;
- (2)  $\omega_+ := d\eta$ ,  $\omega_- := d(J\eta)$  are  $(1, 1)$ -forms on  $H$  and
- (3)  $T_0$  is a trace-free  $(2, 1) + (1, 2)$ -form on  $H$ .

Moreover, if we denote by  $A_+, A_-$  the corresponding  $J$ -invariant skew-symmetric endomorphisms of  $H$ , then

$$[A_+, A_-] = 0 \text{ and } A_\pm.T_0 = 0,$$

where  $A.\alpha(X_1, \dots, X_k) := -\sum_i \alpha(\dots, AX_i, \dots)$ .

*Proof.* The decomposition (26) follows directly from the parallelism of  $\theta$  and of the splitting  $TM = E \oplus H$ . It is also clear that

$$d\eta = T(\eta, \cdot, \cdot),$$

for the  $\nabla$ -parallel form  $\eta$ , which implies that  $\omega_+ := d\eta$  and  $\omega_- := d(J\eta)$ . It remains to show that they are  $(1, 1)$ -forms, the commutation relations, and that  $\eta, J\eta$  are Killing and real-holomorphic (i.e., their flow preserves  $J$ ).

That  $\eta$  and  $J\eta$  are Killing follow from the fact that they are  $\nabla$ -parallel and the torsion  $T$ , relating  $\nabla$  to the Levi-Civita connection  $\nabla^g$ , is skew-symmetric, thus  $\nabla^g\eta$  and  $\nabla^g(J\eta)$  are skew-symmetric endomorphisms of  $TM$ , identified with some multiple of  $A_+$ , resp.  $A_-$ . That their flow preserves  $J$  is thus equivalent to the  $J$ -invariance of  $A_\pm$ , or to  $\omega_\pm$  being of type  $(1, 1)$ .

To prove this, we use Proposition 3.4. As we have seen before,

$$\delta T = 0.$$

On the other hand,

$$\delta T = - * d * T,$$

and we want to use the expression of  $*$  related to the operators induced by  $J$ , as given in Proposition 2.2. For this, we write

$$T = \eta \wedge \omega - \mathcal{J}\Omega_0, \quad (27)$$

with  $\Omega_0 \in \Lambda_0^{(2,1)+(1,2)}M$ , and we get

$$*(-\mathcal{J}\Omega_0) = \Omega_0 \wedge \frac{\omega^{m-3}}{(m-3)!},$$

and

$$*(\eta \wedge \omega) = J\eta \wedge \frac{\omega^{m-2}}{(m-2)!} = \frac{1}{m-2}(J\eta \wedge \omega) \wedge \frac{\omega^{m-3}}{(m-3)!},$$

and recall that  $d\omega = \mathcal{J}T = J\eta \wedge \omega + \Omega_0$ . We obtain

$$*T = d\omega \wedge \frac{\omega^{m-3}}{(m-3)!} - \frac{m-3}{m-2}(J\eta \wedge \omega) \wedge \frac{\omega^{m-3}}{(m-3)!}.$$

We know that  $*T$  is closed, so we have that

$$d\left((J\eta \wedge \omega) \wedge \frac{\omega^{m-3}}{(m-3)!}\right) = 0.$$

But this means that

$$- * d * (\eta \wedge \omega) = 0,$$

so both components of  $T$  in (27) have vanishing codifferential. They are also  $\nabla$ -parallel, so  $\delta^\nabla(\eta \wedge \omega) = 0$  as well. Applying Proposition 3.4, we get that

$$\alpha := \sum_{i < j} (\varepsilon_i \lrcorner \varepsilon_j \lrcorner (\eta \wedge \omega)) \wedge (\varepsilon_i \lrcorner \varepsilon_j \lrcorner T) = 0.$$

But

$$(\eta \wedge \omega)(\varepsilon_i, \varepsilon_j, X) = -\eta(\varepsilon_i)g(\varepsilon_j, JX) + \eta(\varepsilon_j)g(\varepsilon_i, JX) + \omega(\varepsilon_i, \varepsilon_j)\eta(X).$$

Suppose that  $J\varepsilon_{2i-1} = \varepsilon_{2i}$  and that  $\varepsilon_1, \varepsilon_2 \in E$  and the rest spans  $H$ . In order to evaluate  $\alpha$  on various arguments  $X, Y$ , we compute first

$$\begin{aligned} \sum_{i,j=1}^{2m} (\eta \wedge \omega)(\varepsilon_i, \varepsilon_j, X) T(\varepsilon_i, \varepsilon_j, Y) &= \\ &= \sum_{i,j=1}^{2m} (\eta(X)\omega(\varepsilon_i, \varepsilon_j) - \eta(\varepsilon_i)g(JX, \varepsilon_j) + \eta(\varepsilon_j)g(\varepsilon_i, JX)) T(\varepsilon_i, \varepsilon_j, Y) \\ &= 2\text{tr}_\omega T(Y)\eta(X) - T(\eta, JX, Y) + T(JX, \eta, Y). \end{aligned} \tag{28}$$

We obtain thus

$$\alpha(X, Y) = (\text{tr}_\omega T \wedge \eta)(X, Y) - T(\eta, JX, Y) - T(\eta, X, JY) = 0.$$

We know that  $\text{tr}_\omega T$  is colinear to  $\eta$ , thus we obtain that  $T(\eta, \cdot, \cdot)$  is a  $(1, 1)$ -form. But this means that  $\omega_+ \in \Lambda(1, 1)M$ , as required. By  $\mathcal{J}T = J.T$  it also follows that  $\omega_- \in \Lambda(1, 1)M$ . Note that this already implies that

$$T(\eta, J\eta, \cdot) = 0. \tag{29}$$

The commutation relations follow from (25); indeed, from  $\nabla\eta = 0$ , it follows that the curvature terms  $R(X, Y, Z, \eta)$  vanish, thus

$$\Omega(X, Y, Z, \eta) = 0, \quad \forall X, Y, Z \in TM.$$

From (24), we obtain, by setting  $Z := J\eta$ ,

$$\begin{aligned} 0 = \Omega(X, Y, J\eta, \eta) &= \sum_{i=1}^{2m} (T(\varepsilon_i, X, \eta)T(\varepsilon_i, Y, J\eta) - T(\varepsilon_i, Y, \eta)T(\varepsilon_i, X, J\eta) - \\ &\quad - T(\varepsilon_i, X, Y)T(\varepsilon_i, \eta, J\eta)) \end{aligned} \tag{30}$$

where the last term vanishes from (29). But this is equivalent to

$$\sum_{i=1}^{2m} \omega_+(\varepsilon_i, \cdot) \wedge \omega_-(\varepsilon_i, \cdot) = 0,$$

and this is equivalent to  $[A_+, A_-] = 0$ .

If we set in (24)  $X, Y, Z \in H$  and  $V := \eta$ , (30) implies, in a similar way, that

$$A_+ \cdot \Omega_0 = 0,$$

and, if  $V = J\eta$ , we obtain  $A_- \cdot \Omega_0 = 0$ , as required.  $\square$

The skew-symmetric endomorphisms  $A_\pm$  can thus be diagonalized simultaneously, thus we decompose  $H$  in the eigenspaces  $H_i$ ,  $i = 1, \dots, k$ , of these endomorphisms, such that

$$A_\pm = a_i^\pm J \text{ on } H_i.$$

Of course, the decomposition

$$TM = E \oplus \bigoplus_{i=1}^k H_i$$

is orthogonal and  $\nabla$ -parallel, thus multiplying the metric  $g$  with some constants on each  $H_i$ , and even changing the complex structure on  $E$  (see below) will produce a new  $\nabla$ -parallel hermitian structure  $(g', J')$ . However, even though the torsion  $T$  of  $\nabla$  is unchanged, the torsion tensor  $g(T(\cdot, \cdot), \cdot)$  will change, and it is not necessarily skew-symmetric any more. The relation between the characteristic torsions of  $(g, J)$  and  $(g', J')$  is investigated in the following

**Proposition 3.6.** *Let  $(M, g, J)$  a hermitian manifold with characteristic connection  $\nabla$  and parallel torsion*

$$T = \eta \wedge \omega_+ + J\eta \wedge \omega_- + T_0.$$

*Denote by  $A_\pm$  the  $J$ -invariant skew-symmetric endomorphisms corresponding to  $\omega_\pm$ . Suppose*

$$TM = E \oplus H = \bigoplus_{i=0}^k H_i \tag{31}$$

*is the  $\nabla$ -parallel and  $g$ -orthogonal decomposition of  $TM$  in  $H_0 = E$ , the complex line generated by the Lee form, and*

$$H := E^\perp = \bigoplus_{i=1}^k H_i,$$

*where  $H_i$  are the eigenspaces of the (commuting) endomorphisms  $A_+$ ,  $A_-$ .*

*Let  $(g', J')$  be a  $\nabla$ -parallel hermitian structure, such that*

$$J'|_H = J|_H, g'|_{H_i} = a_i g|_{H_i}, \quad i = 1, \dots, k,$$

*and such that the decomposition (31) is also  $g'$ -orthogonal.*

*Then the following hold:*

- (1)  $J'$  is integrable,
- (2) the characteristic connection  $\nabla'$ , corresponding to the hermitian structure  $(g', J')$  differs from  $\nabla$  by a  $\nabla$ -parallel tensor  $A : TM \rightarrow \text{End}(TM)$ ,  
Suppose now that  $(M, g, J)$  is LP. Then we also have
- (3) for any  $X \in TM$ ,  $A_X$  is skew-symmetric w.r.t. both metrics  $g, g'$ , and it commutes with both complex structures  $J, J'$ .
- (4) The Lee form of  $\omega' := g'(J'\cdot, \cdot)$  is a linear combination (with constant coefficients) of  $\xi$  and  $J\xi$ , where  $\xi$  is the Lee form of  $\omega := g(J\cdot, \cdot)$
- (5) the torsion  $T'$  of  $\nabla'$  is parallel w.r.t. both connections  $\nabla, \nabla'$ ,
- (6)  $\xi$  and  $J\xi$  are  $\nabla'$ -parallel as well,

*Proof.* The first claim follows from that fact that all  $\nabla$ -parallel sections of  $E$  are (real parts of) holomorphic vector fields (both for  $J$  and  $J'$ ). Indeed, we consider the Nijenhuis tensor  $N'$  of  $J'$ :

$$4N'(X, Y) = [J'X, J'Y] - J'[J'X, Y] - J'[X, J'Y] - [X, Y],$$

and we consider the cases  $X, Y \in E$ ,  $X, Y \in H$  and  $X \in E$ ,  $Y \in H$ . In the first case  $N'(X, Y) = 0$  holds because  $X, Y$  are tangent to some 2-dimensional leaves, and every almost complex structure on such a manifold is integrable, and in the second case, we can replace  $J'$  by  $J$ .

In the third case, we have  $J'Y = JY$  and, because  $X$  and  $J'X$  are Killing, they preserve the distribution  $H$ , thus all  $J'$ 's can be replaced with  $J$ s except for the two occurrences of  $J'X$ . The Nijenhuis tensor becomes, in this case,

$$4N'(X, Y) = \mathcal{L}_{J'X}J(Y) - J\mathcal{L}_X(Y),$$

and both terms vanish because the flows of both  $X$  and  $J'X$  preserve  $J$ .

This proves claim 1.

For claim 2, we know that, because  $J'$  is integrable, there exists a (unique) characteristic connection  $\nabla'_X Y = \nabla_X Y + A_X Y$  for the hermitian structure  $(g', J')$ , [1]. In what follows, we compute it explicitly:

Denote by

$$\begin{aligned} A(X, Y, Z) &:= g'(A_X Y, Z), \\ \tau(X, Y, Z) &:= g'(T(X, Y), Z), \\ T'(X, Y, Z) &:= g'(T'(X, Y), Z), \end{aligned}$$

where  $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$  and  $T'(X, Y) = \nabla'_X Y - \nabla'_Y X - [X, Y]$  are the torsions of  $\nabla$ , resp.  $\nabla'$ . The conditions  $\nabla'g' = 0$ ,  $\nabla'J' = 0$  and  $T' \in \Lambda^3 M$  imply:

$$A(X, Y, Z) + A(X, Z, Y) = 0, \tag{32}$$

$$A(X, J'Y, J'Z) = A(X, Y, Z), \text{ and, resp.} \tag{33}$$

$$A(Y, X, Z) + A(Z, X, Y) = \tau(X, Y, Z) + \tau(X, Z, Y), \tag{34}$$

for all vectors  $X, Y, Z \in TM$ . These equations imply, after some computations,

$$\begin{aligned} A(X, Y, Z) = \frac{1}{2} \bigg( & (-\tau(X, Y, Z) + \tau(J'X, J'Y, Z)) + \\ & + (\tau(Y, Z, X) + \tau(J'Y, J'Z, X)) - \\ & - (\tau(Z, J'X, J'Y) + \tau(J'Z, X, J'Y)) \bigg). \end{aligned} \tag{35}$$

Therefore,  $A$  is  $\nabla$ -parallel (claim 2.), but, strictly speaking, we haven't proven that the equations (35) imply the relations (32-34). This implication follows by the existence of the characteristic connection [1], or, alternatively, by the following lemma:

**Lemma 3.7.** *Let  $(M, g, J, \nabla)$  as above, and let  $(g', J')$  be a  $\nabla$ -parallel hermitian structure on  $M$  (thus  $J'$  is supposed integrable). Then the tensor  $A$  defined in (35) satisfies the relations (32-34).*

*Proof.* This follows by straightforward computation; each of the equations (32-34) turns out to be equivalent to the fact that the Nijenhuis tensor  $N'$  becomes totally skew-symmetric if we use the metric  $g'$ , which follows from the integrability of  $J'$ . (Note that  $g(N'(\cdot, \cdot), \cdot)$  is always totally skew-symmetric, because this tensor can be expressed, for the  $\nabla$ -parallel  $J'$ , in terms

of the torsion  $T$ , which is totally skew-symmetric; however, this does not necessarily imply that  $g'(N'(\cdot, \cdot), \cdot) \in \Lambda^3 M$ , which is what we want.)  $\square$

We can thus assume that the connection  $\nabla'$  is indeed the characteristic connection of the hermitian structure  $(g', J')$ , and that it differs from  $\nabla$  by a  $\nabla$ -parallel tensor  $A$  (claim 2.). let us compute now the values of  $A$  on specific vectors. First, let  $X \in H_i$ ,  $Y \in H_j$  and  $Z \in H_l$ , for  $i, j, l = 1, \dots, k$ . Then  $J'$  coincides with  $J$  on these vectors, and the  $g'$  scalar product with the elements  $Z$ ,  $X$  and  $J'Y = JY$  can be replaced, in (35), by  $g$  multiplied by the factors  $a_l$ ,  $a_i$  and, resp.  $a_j$ . We replace thus  $\tau$  by some multiples of  $T$  and we obtain:

$$\begin{aligned} 2A(X, Y, Z) &= 2a_l g(A_X Y, Z) = \\ &= (a_i - a_l)T(X, Y, Z) + (a_l - a_j)T(JX, JY, Z) + (a_i - a_j)T(X, JY, JZ), \quad (36) \\ \forall X \in H_i, Y \in H_j, Z \in H_l. \end{aligned}$$

Of course, the  $J$ -invariance of  $A_X$  is obvious for these arguments, but if we want to check that  $g(A_X Y, Z) + g(A_X Z, Y) = 0$ , we need to check that

$$\frac{2}{a_l}A(X, Y, Z) + \frac{2}{a_j}A(X, Z, Y) = 0.$$

We use (36) and note that the first and the third terms on the right hand side are already skew-symmetric in  $Y$  and  $Z$ . We obtain thus

$$\frac{2}{a_l}A(X, Y, Z) + \frac{2}{a_j}A(X, Z, Y) = \frac{a_l - a_j}{a_l}T(JX, JY, Z) + \frac{a_j - a_l}{a_j}T(JX, JZ, Y).$$

This vanishes for any positive numbers  $a_1, \dots, a_k$  iff  $T_0(X, Y, Z) = 0$  each time two of the vectors  $X, Y, Z$  belong to two different eigenspaces  $H_i \neq H_j$ . This is satisfied by the hypothesis that  $(M, g, J)$  is LP, i.e.,  $T_0 = 0$ . But in this case

$$A(X, Y, Z) = 0, \quad \forall X \in H_i, Y \in H_j, Z \in H_l. \quad (37)$$

The other case is when  $X, Y$  or  $Z$  belongs to  $E$ . (If two of them belong to  $E$ , all the terms in (35) vanish). Let  $Z \in E$ ,  $X \in H_i$  and  $Y \in H_j$ , with  $i, j \geq 1$ . We replace  $J'X$  and  $J'Y$  by  $JX$ , resp.  $JY$ , and the first line of the right hand side of (35) vanishes. We obtain thus

$$\begin{aligned} 2A(X, Y, Z) &= a_i(T(Y, Z, X) + T(JY, J'Z, X)) - a_j(T(Z, JX, JY) + T(JZ, X, JY)), \quad (38) \\ \forall X \in H_i, Y \in H_j. \end{aligned}$$

We use again that  $T(Z, \cdot, \cdot)$  is  $J$ -invariant for any  $Z \in E$ , thus

$$2A(X, Y, Z) = (a_i - a_j)(T(X, JY, J'Z) + T(X, Y, Z)).$$

Recall that  $T(X, Y, Z) = 0$ ,  $\forall Z \in E$  and  $X, Y$  eigenvectors of  $A_\pm$  for *different* eigenvalues. Thus  $A(X, Y, Z) = 0$  if  $i \neq j$ , therefore

$$A(X, Y, Z) = 0 \quad \forall Z \in E.$$

Because of the skew-symmetry of  $A$  in the last two arguments, it follows equally that

$$A(X, Y, Z) = 0 \quad \forall Y \in E.$$

It remains thus to compute  $A(X, Y, Z)$  with  $X \in E$ . Let  $Y \in H_j$  and  $Z \in H_l$ ,  $j, l \geq 1$ . Using (35), after some similar computations, we get

$$\begin{aligned} A(X, Y, Z) &= a_l g(A_X Y, Z) = \left( g' - \frac{a_j + a_k}{2} \right) (T(Y, Z), X) + \frac{a_l - a_j}{2} T(J'X, JY, Z), \quad (39) \\ \forall X \in E, Y \in H_j, Z \in H_l. \end{aligned}$$

The  $J$ -invariance of  $A_X$ , for  $X \in E$ , follows directly, so it remains to check that  $A_X$  is also  $g$ -skew-symmetric. Note first that, as we saw earlier, the terms  $T(Y, Z)$  and  $T(JY, Z)$  have no component in  $E$  if  $Y$  and  $Z$  belong to different eigenspaces  $H_j \neq H_l$  of  $A_\pm$ . Thus

$$A_X Y \in H_j, \quad \forall X \in E, \quad Y \in H_j. \quad (40)$$

We suppose thus  $Y, Z \in H_j$  and we obtain

$$g(A_X Y, Z) = \left( \frac{2}{a_j} g' - 2g \right) (T(Y, Z), Z),$$

which is skew-symmetric in  $Y, Z$ . This proves Claim 3.

We have shown (using the LP condition) that  $A_X Y = 0 \quad \forall X, Y \in H$ . Therefore,

$$T'(X, Y, Z) = g'(T(X, Y), Z), \quad \forall X, Y \in H,$$

moreover  $T'(X, Y, Z) = 0$  if  $X, Y, Z \in H$  (because  $M$  is LP). Therefore, the trace of  $T'$  w.r.t.  $\omega'$  is in the dual space to  $E$ , which proves the fourth claim. Note that it is possible that this trace vanishes for some choice of the constants  $a_1, \dots, a_k$ .

Because  $A$  (and  $T$ ) is algebraically expressed by  $A_\pm$  and  $\xi$  and  $J\xi$ , and because  $[A_+, A_-] = 0$ , it follows that  $A_X A = 0$  and  $A_X T = 0$ ,  $\forall X \in E$ , therefore

$$\nabla'_X A = \nabla_X A + A_X A = 0$$

and then also  $\nabla'_X T = 0$ ,  $\forall X \in E$ . For  $X \in H$   $\nabla'_X = \nabla_X$ , therefore  $A$ ,  $T$  and all linear combinations (with constant coefficients) of  $\xi$  and  $J\xi$  are  $\nabla'$ -parallel as well (Claims 5 and 6).  $\square$

**Definition 3.8.** *The modification  $(g', J')$  of the GCE structure  $(g, J)$ , as in the Proposition 3.6, is called parallel modification. A curve  $t \mapsto (g_t, J_t)$ , with  $(g_0, J_0) = (g, J)$ , of parallel modified GCE structures is called a parallel deformation of GCE structures.*

**Remark 3.9.** *The  $\nabla$ -parallelism of  $J'$ , for a hermitian structure with parallel characteristic torsion, does not even imply, in general, that it is integrable, as an example from twistor theory shows [7]: indeed, for a compact nearly Kähler 6-manifold  $(M, g, J)$  with reduced characteristic holonomy (i.e., there is a  $\nabla$ -invariant splitting  $TM = E \oplus H$  in complex subbundles), it follows that  $(M, g, J)$  is the twistor space of  $S^4$  or of  $\hat{\gamma}^2$ , for the non-integrable almost complex structure  $J$  [7]. By exchanging  $J$  by  $-J$  on the twistor fibers, we obtain an integrable (in fact, it is even Kähler w.r.t. to a rescaled metric  $g'$ ) complex structure on  $M$ . If we do not change the metric  $g$ , the characteristic connection  $\nabla$  for the hermitian structure  $(g, J)$  is still characteristic for  $(g, J')$ , and its torsion is still parallel (for a nearly Kähler manifold  $(M, g, J)$ , the characteristic torsion is always parallel, [11], [7]). The  $\nabla$ -parallel modification  $J$  of  $J'$  is, of course, not integrable. The point here is that  $(M, g, J')$  is hermitian, but with vanishing Lee form. It is thus essential that the modification of the hermitian structure  $(g, J)$  is made regarding of the Lee distribution  $E$ , in order to obtain the results in Proposition 3.6.*

We can use the modification  $(g', J')$  of the hermitian GCE metric  $(g, J)$  to prove that the Calabi-Eckmann complex structures  $J_a$  admit GCE hermitian structures (Proposition 3.2). Note that the Lee form doesn't change if all  $a_i = 1$  (even if  $J' \neq J$ ).

Another application of Proposition 3.6 is to determine the local structure of a GCE manifold provided  $k \leq 2$ , i.e., there are at most two common eigenspaces of  $A_\pm$ ; the case where there is only one eigenspace  $H$  is the Vaisman case: indeed, in this case  $\omega_- = 0$  and  $\omega_+$  must be a multiple of  $\omega|_H$ . If  $k = 2$ , the following cases occur:



**Corollary 3.10.** *Let  $H := H_1 \oplus H_2$  be the (non-trivial) decomposition of  $H = E^\perp$  in common eigenspaces of the endomorphisms  $-J \circ A_\pm$  corresponding to the exact  $(1,1)$ -forms  $\omega_+ = d\eta$  and  $\omega_- = d(J\eta)$  in Proposition 3.6, and let  $a_1^\pm \geq a_2^\pm$  be the corresponding eigenvalues. Then the following situations occur:*

- (1)  $\omega_- = 0$ ,  $a_2^+ < 0 < a_1^+$ ;
- (2)  $\omega_- = 0$ ,  $a_2^+ = 0 < a_1^+$ ;
- (3)  $\omega_- = 0$ ,  $0 < a_2^+ \leq a_1^+$ ;
- (4)  $\omega_- \neq 0$ .

Then, by an appropriate choice of hermitian structure  $(g', J')$  as in Proposition 3.6, we obtain the following structures:

- (1) In cases 1. and 3.,  $(M, g', J')$  is a (pseudo)-Vaisman manifold (In case 1., the metric has mixed signature: it is negative-definite on  $H_2$ );
- (2) In case 2.,  $(M, g', J')$  is locally the product of a Sasakian manifold, a real line and a Kähler manifold;
- (3) In the (generic) case 4.,  $(M, g', J')$  is locally the product of two Sasaki manifolds.

*Proof.* If  $\omega_- = 0$ , then the constants  $a_i$  will be chosen such that the eigenvalues of  $\omega_+$  are  $-1, 0$  or  $1$  w.r.t.  $\omega'$  (we put  $J' := J$ ). In this case, the metric  $g'$  (and actually  $g$  itself) clearly has closed Lee form. In Case 3., the metric is already Vaisman, and in case 1., the pseudo-hermitian metric  $\omega^s := \omega'|_E + \omega'|_{H_1} - \omega'|_{H_2}$  is locally conformally pseudo-Kähler with parallel Lee form, therefore pseudo-Vaisman.

If  $\omega_- \neq 0$ , then  $\alpha_1 > 0 > \alpha_2$ , since the trace of  $\omega_-$  w.r.t.  $\omega$  is zero. On the other hand, the eigenvalues  $a_1^+, a_2^+$  are not both zero, thus the vectors  $a^+ := (a_1^+, a_2^+)$  and  $a^- := (a_1^-, a_2^-)$  form a basis in  $\mathbb{R}^2$ . There exists a unique invertible  $2 \times 2$  real matrix

$$R := \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

such that  $R(2, 0) = a^+ \in \mathbb{R}^2$  and  $R(0, 2) = a^-$ . We let  $R$  act on  $E \simeq \mathbb{R}^2$  set  $\eta' : R\eta$  and  $J'\eta' := R(J\eta)$ . This defines the complex structure  $J'$  and we set  $g'$  on  $E$  such that  $\eta'$  and  $J'\eta'$  are unit vectors. We can write then

$$T'(\eta', \cdot, \cdot) = d\eta' = d(R\eta) = 2\omega|_{H_1}$$

and

$$T'(J'\eta', \cdot, \cdot) = d(J'\eta') = d(R(J\eta)) = 2\omega|_{H_2},$$

therefore the distributions  $D_1 := \mathbb{R}\eta' \oplus H_1$  and  $D_2 := \mathbb{R}J'\eta' \oplus H_2$  are both integrable and the leaves of the corresponding foliation are Sasakian. These foliations are  $g'$ -orthogonal to each other and correspond to a local product structure.  $\square$

**Remark 3.11.** *There is another way to reduce case 1. from the above Corollary to a Vaisman manifold (for a positive-definite metric, but for the opposite orientation): we can replace  $J$  on  $H_2$  by  $J' := -J$  and let  $J' = J$  on  $H_1$  and on  $E$ ; the corresponding Kähler form changes sign on  $H_2$  and we obtain a GCE hermitian metric (the integrability of this  $J'$  is easily checked in this GCE setting — it would not necessarily be true in general, cf. Remark 3.9 above)  $(g, J')$  corresponding to the case 3. of the Corollary.*

Conversely, the product of two Sasakian manifolds is GCE with the product metric  $G$  and the corresponding complex structure  $J$ , but also w.r.t. the *parallel modification*  $(g', J')$  of the

hermitian structure  $(g, J)$  from Proposition 3.6. We will call these GCE manifolds *modified Sasakian products*.

#### 4. COMPACT GENERALIZED CALABI-ECKMANN MANIFOLDS OF DIMENSION 6

In this section,  $(M^6, J, g)$  denotes a compact GCE hermitian manifold. The torsion  $T$  of the characteristic connection  $\nabla$  is parallel and decomposes according to Proposition 3.5 (with  $T_0 = 0$ ). Of course, the number of common eigenspaces to  $A_{\pm}$  is at most 2, because  $H$  is complex 2-dimensional. We have seen in Corollary 3.10 that a parallel change of hermitian structure reduces a GCE structure either to a Vaisman threefold (if  $\partial(J\xi) = 0$ ) or to a local Sasakian product (in the generic case).

In the generic case, we will give a full description of the complex manifold  $(M, J)$  and of the corresponding hermitian metric  $g$ .

We can assume, possibly after a parallel modification, that a GCE metric of generic type (i.e.,  $\omega_- \neq 0$ ), has as universal covering  $(\tilde{M}, g, J)$  a Riemannian product of two Sasakian 3-manifolds  $(N_i, g_i, \xi_i)$ ,  $i = 1, 2$ , where  $\xi_i$  are the corresponding unit Reeb vector fields.

**Remark 4.1.** *A Sasakian structure can be defined in many ways:*

- (1) *A Riemannian metric, with a unit Killing vector field of certain type, plus an integrability condition;*
- (2) *A CR-structure admitting a Reeb vector field whose flow preserves it;*
- (3) *A non-vanishing vector field  $\xi$ , a complex (integrable) structure on the quotient bundle  $TM/\mathbb{R}\xi$  which is invariant to the flow of  $\xi$ , and a compatible dual 1-form to  $\xi$ .*

The latter is the one that we will use here.

**Definition 4.2.** *A pre-Sasakian structure on a 3-manifold  $M$  is a non-vanishing vector field  $\xi$  and a complex structure  $J$  on  $TM/\mathbb{R}\xi$  that is invariant by the flow of  $\xi$ .*

**Proposition 4.3.** *A Sasakian structure on  $M$  is given by a pre-Sasakian structure together with a  $\xi$ -invariant contact form  $\eta$  such that*

$$\eta(\xi) = 1,$$

*and that  $d\eta(\cdot, J\cdot)$  is symmetric and positive definite on  $TM/\mathbb{R}\xi$ .*

*Proof.* Indeed, the Sasakian metric is then

$$g := \eta^2 + \frac{1}{2}d\eta(J\cdot, \cdot).$$

□

Recall that in the classification of Sasakian structures on compact 3-manifolds [4], the first geometrical information that we were able to give was regarding the *pre-Sasakian structure*, in particular on the Reeb vector field  $\xi$ . Once the Sasakian structure was given, it was possible to deform the contact form, by keeping the same pre-Sasakian structure, in order to get a locally homogeneous metric (this deformation of the CR structure was called *of second type*, while the *deformations of the first type* consisted in changing the metric while keeping the same underlying CR structure; globally, these deformations only occur on the sphere  $S^3$  or some of its quotients, [4], [5], [6]).

By analogy we define:

**Definition 4.4.** An elliptic pre-complex structure on a six-dimensional manifold  $M$  is a free action of  $\mathbb{C}$  on  $M$ , generating an integrable distribution by 2-planes  $D$ , and a  $\mathbb{C}$ -invariant complex integrable structure  $J^D$  on  $TM/D$ , i.e.,  $(J^D)^2 = -I$  on  $TM/D$ .  $J^D$  is called integrable if the Nijenhuis tensor

$$4N^{J^D}(X, Y) := [J^D X, J^D Y] - J^D[J^D X, Y] - J^D[X, J^D Y] - (J^D)^2[X, Y]$$

has values in  $D$ , for any vector fields  $X, Y$  on  $M$ . Here,  $J^D$  is extended to an endomorphism of  $TM$  by being zero on  $D$ .

In case  $M$  is the total space of an elliptic holomorphic fibration over a complex manifold, the *elliptic pre-complex structure* contains the information regarding the elliptic fibers, the fibration itself (in the smooth category), and the complex structure on the base manifold. It is not clear if, for a given such structure, a compatible complex structure on  $M$  exists, inducing the given elliptic pre-complex structure. In this paper, the considered elliptic pre-complex structures are always induced by some integrable complex structures.

We will show that, if  $(M^6, J, g)$  is a compact, generalized Calabi-Eckmann hermitian manifold of generic type (i.e., it is locally isometric to a *modified Sasakian product*, see above), then its elliptic pre-complex structure is locally homogeneous:

**Theorem 4.5.** Let  $(M^6, J, g)$  be a compact, generalized Calabi-Eckmann hermitian manifold of generic type and suppose (after parallel modification) that its universal covering  $\tilde{M}$  is isometric to a Riemannian product of two Sasakian 3-manifolds  $M_1$  and  $M_2$ . Denote by  $\xi_i$ ,  $i = 1, 2$ , the Reeb vector fields of these Sasakian structures, and let  $E := \mathbb{R}\xi_1 \oplus \mathbb{R}\xi_2$ , and  $J^E$  be the induced complex structure on  $TM/E$  (this complex structure is  $\xi_i$ -invariant,  $i = 1, 2$ ).

Then there is another hermitian structure  $(J', g')$  on  $M$ , which is obtained from  $(J, g)$  through an isotopy of generalized Calabi-Eckmann structures of generic type, and such that  $(\tilde{M}^6, J', g')$  is a Lie group  $G$  and  $(J', g')$  is a left-invariant hermitian structure. If both  $M_1, M_2$  are non-compact, then the isotopy can be chosen to preserve the pre-elliptic complex structure.

Moreover, the Lie group  $G$  is the product  $G = G_1 \times G_2$ , where  $G_i \in \{SU(2), \widetilde{SL}(2, \mathbb{R}), Nil_3\}$ , and the fundamental group  $\pi_1(M)$  acts on  $G$  by automorphisms of the Sasakian product (this group is 8-dimensional).

**Remark 4.6.** In some cases (if one of the factors  $G_i$  is compact), we can show that, after passing to a finite covering, the fundamental group  $\pi_1(M)$  is actually a cocompact lattice of  $G$ , which is a subgroup of codimension 2 in the group of automorphisms of the structure, but it is unknown whether this fact holds in general.

*Proof.* Consider  $\tilde{M}$  the universal covering of  $M$ , and lift the metric  $g$  to  $\tilde{M}$ , which becomes a complete Riemannian manifold, on which the de Rham decomposition theorem implies

$$(\tilde{M}, g) = (M_1, g_1) \times (M_2, g_2),$$

where  $(M_i, g_i)$ ,  $i = 1, 2$  are complete, *cocompact* Sasakian 3-manifolds.

Here, *cocompact* means that there exists a compact set  $K_i \subset M_i$  and a group of isometries  $\Gamma_i$  of  $M_i$  such that

$$M_i = \bigcup_{\gamma \in \Gamma_i} \gamma K_i,$$

i.e., the whole manifold can be reconstructed by the action of the group  $\Gamma_i$  of isometries, by replicating a compact set  $K_i$ .

Note that  $\tilde{M}$  is cocompact with respect to  $\pi_1(M)$ , and thus  $M_i$  is cocompact w.r.t.  $\Gamma_i$ , the projection of  $\pi_1(M)$  on the factor  $M_i$  in the de Rham decomposition of  $\tilde{M}$ . Because  $\tilde{M}$  is simply connected, so are its factors  $M_i$ ,  $i = 1, 2$

We want to characterize the Sasakian structures on  $M_i$ ,  $i = 1, 2$ . We have two situations:

- (1)  $M_i$  is compact (and simply connected), hence  $(M_i, g_i)$  is one of the Sasakian structures on the sphere  $S^3$ , see [4],[6].
- (2)  $M_i$  is not compact, but still cocompact.

In the second case we cannot apply the results about the Sasakian structures on *compact* 3-manifolds in [4], [5], [6], but we will retrieve them.

Consider  $P_i$  the circle bundle over  $M_i$  consisting of the unit vectors in the complex distribution  $H_i$ , orthogonal to the Reeb vector field  $\xi_i$ . Therefore, we have the following principal fiber bundle:

$$S^1 \rightarrow P_i \rightarrow M_i. \quad (41)$$

The group  $G_i$  (containing  $\Gamma_i$ ) of Sasakian automorphisms of  $M_i$ , i.e., the group of isometries of  $(M_i, g_i)$  preserving  $\xi_i$ , can thus be realized as the group of automorphisms of the canonical parallelization of  $P_i$  [12], [3].  $G_i$  is thus a Lie group and it can be realized as a closed (embedded) submanifold of  $P_i$  by any of its orbits, [3], [12]. Moreover, the action of  $G_i$  on  $P_i$  is free and proper, thus  $G_i \backslash P_i$  is a manifold, basis of a  $G_i$ -principal bundle (of which the total space is  $P_i$ ).

It is equally well-known [3], [12], that a subgroup of  $G_i$  is closed iff its orbits in  $P_i$  are closed. This, in return, is equivalent to its orbits in  $M_i$  being closed, because the fibers of  $P_i$  over  $M_i$  are compact.

We consider the group  $\Phi_i$  generated by the flow of  $\xi_i$ . The closure of  $\Phi_i$  in  $G_i$  is either

- (1)  $\Phi_i$  itself,  $\Phi_i$  being then a circle or a real line;
- (2) an abelian group of dimension at least 2

We begin with the second case: as the only abelian Lie groups that admit a dense subgroup of dimension 1 are tori, we conclude that we have the following principal fibration, with fiber  $T^k$ , the  $k$ -dimensional torus that is the closure in  $G_i$  of  $\Phi_i$ :

$$T^k \rightarrow P_i \rightarrow S_i, \quad (42)$$

where  $S_i$  is the  $4 - k$ -dimensional quotient of  $P_i$  by the torus  $T_i$ . Note that by assumption  $P_i$  is not compact, so neither is  $S_i$ . This already excludes the case  $k = 4$ , so we have only the cases  $k = 2$  ( $S_i$  is a non-compact oriented surface) or  $k = 3$  ( $S_i$  is a real line). In both cases  $\pi_2(S_i) = 0$ .

We write now the long homotopy sequences corresponding to the fibrations (41) and (42):

$$\dots \rightarrow 0 = \pi_2(S_i) \rightarrow \pi_1(T^k) \rightarrow \pi_1(P_i) \rightarrow \dots;$$

$$\dots \rightarrow \pi_1(S^1) \rightarrow \pi_1(P_i) \rightarrow \pi_1(M_i) = 0.$$

From the second line we obtain that  $\pi_1(P_i)$  is a quotient of  $\mathbb{Z}$ , but the first line implies that it must contain at least  $\mathbb{Z}^k$ ,  $k \geq 2$ , contradiction.

We have thus shown:

**Lemma 4.7.** *On a complete non-compact, simply connected Sasakian 3-manifold, the flow of the Reeb vector field is closed in the group of Sasakian automorphisms, and acts properly on the manifold.*

We have thus to distinguish between two cases

- (1)  $\Phi_i \simeq \mathbb{R}$
- (2)  $\Phi_i \simeq S^1$ .

In the first case,  $\Phi_i$  acts freely on  $P_i$ , but not necessarily on  $M_i$ . In the second case, the periods of the orbits of  $\Phi_i$  on  $M_i$  could be, in principle, a quotient of the period of  $\Phi_i \simeq S^1$  by a natural number. The following lemma shows that this does not happen:

**Lemma 4.8.** *Let  $M_i$  be a non-compact, cocompact (w.r.t. the Sasakian automorphism group) complete Sasakian 3-manifold. Then if  $\phi_t$  has a fixed point, then it is the identity.*

*Proof.* Let  $\gamma := \phi_t$  and suppose  $\gamma(x) = x$ . Because  $\gamma$  is an isometry that fixes  $x$ , it is the identity iff its derivative at  $x$  is the identity. Of course,  $\gamma_*\xi = \xi$ , so  $\gamma_*$  has to be a rotation in the plane  $H_i$  at  $x$ . If  $\gamma_* \neq I$ , then  $\gamma(y) \neq y$  for any  $y$  close to  $x$ , but not on the orbit of  $\Phi_i$  through  $x$ . In other words, if  $\gamma$  is not the identity, the neighboring orbits of  $\Phi_i x$  do not close at time  $t$ .

We know, on the other hand, that, although  $M_i$  is itself not compact, there is a compact set  $K_i$ , sufficiently large, such that  $M_i$  is the (infinite) union of sets isometric to  $K_i$ . If we consider  $K_i$  to contain the closed orbit  $\Phi_i x$ , we conclude that there are (infinitely many, hence) at least two at time  $t$  closed orbits of  $\Phi_i$  on  $M_i$ , say  $\Phi_i x \neq \Phi_i y$ . But these two circles can be joined by a minimizing geodesic in the complete manifold  $M_i$ , and this minimizing geodesic must be orthogonal to both circles. The flow of  $\xi$  generates, thus, a set of such minimizing geodesics between  $\Phi_i x$  and  $\Phi_i y$ . It turns out that every point on such a geodesic is a fixed point for  $\gamma = \phi_t$ . In conclusion all the points on a geodesic through  $x$ , orthogonal to  $\xi_x$ , are fixed by  $\gamma$ , contradiction.  $\square$

We apply the lemma as follows: if  $\phi_t$  has a fixed point, then  $\Phi_i$  is  $t$ -periodic. This rules out the case when  $\Phi_i \simeq \mathbb{R}$  acts properly, but not freely on  $M_i$ . If  $\Phi_i$  is a circle, then the Lemma shows that the period of  $\Phi_i$  is the minimal  $t > 0$  such that  $\phi_t$  has a fixed point on  $M_i$ . Therefore,  $\Phi_i$  acts freely on  $M_i$  in this case as well.

We have shown:

**Lemma 4.9.** *On a complete, non-compact, cocompact, simply-connected Sasakian 3-manifold, the Reeb orbits form a principal fiber bundle over a Riemann surface  $S_i$ .*

We want now to show that the Riemann surface is non-compact and simply connected.

For this, we use again the long homotopy sequence for the fibration

$$\Phi_i \rightarrow M_i \rightarrow S_i,$$

and obtain

$$\dots \rightarrow \pi_2(S_i) \rightarrow \pi_1(\Phi_i) \rightarrow \pi_1(M_i) = 0.$$

If  $S_i$  is not the sphere, then  $\pi_2(S_i) = 0$ , hence  $\pi_1(\Phi_i) = 0$  and  $\Phi_i$  is a real line.

On the other hand, the same homotopy sequence shows that  $\pi_1(S_i) \simeq \pi_0(\Phi_i) = 0$ , so  $S_i$  is simply connected. The metric on  $M_i$  induces a metric on  $S_i$ , in particular  $S_i$  is a simply-connected complex curve, therefore

- (1)  $S_i \simeq S^2$  and  $\text{Isom}(S_i) \subset SO(3)$ ;
- (2)  $S_i \simeq \mathbb{C}$  and  $\text{Isom}(S_i) \subset \mathbb{C} \rtimes \mathbb{C}^*$ , the group of complex automorphisms of  $\mathbb{C}$ ;
- (3)  $S_i \simeq \mathbf{H} := \{z \in \mathbb{C} \mid \text{Im} z > 0\}$ , the upper half-plane in  $\mathbb{C}$  and  $\text{Isom}(S_i) \subset PSL(2, \mathbb{R})$  the group of complex automorphisms of  $\mathbf{H}$ .

We want to exclude the case when  $S_i \simeq S^2$ ; indeed, if this holds, then  $\Phi_i$  must be a line, otherwise  $M_i$  is compact. But then,  $M_i$  is a trivial bundle over the sphere, and this cannot have a Sasakian structure adapted to the fibration since the connection in this principal bundle has a non-exact curvature (a multiple of the volume form of  $S_i$ ), contradiction.

Therefore  $S_i$  is either biholomorphic to  $\mathbb{C}$  or to  $\mathbf{H}$ . Since both are contractible,  $M_i$  is homotopically equivalent to  $\Phi_i$ , thus the latter has to be simply connected, hence isomorphic to  $\mathbb{R}$ .

We recall that  $M_i$  admits a cocompact action by Sasakian automorphisms. Therefore, the Riemann surface  $S_i$  admits a cocompact action by isometries, in particular by biholomorphisms. Let us denote by  $\Gamma_i^S$  the group of (induced) biholomorphisms of  $S_i$ .

**Lemma 4.10.** *Let  $N \rightarrow S$  be an  $\mathbb{R}$  principal bundle over a contractible Riemann surface  $S$ , endowed with a connection form  $\lambda \in \Lambda^1(N)$  whose curvature  $d\lambda$  is minus twice the Kähler form  $\omega$  on  $S$ . Then every isometry of  $S$  admits a lift as a Sasakian automorphism of  $N$ .*

*Proof.*  $S$  being contractible, every bundle is topologically trivial. Let  $(t, z) \in \mathbb{R} \times \mathbb{C}$  be coordinates on  $N$  such that  $\xi = \partial_t$  and  $\omega$  the Kähler form on  $S$ . Let  $\lambda = dt + d\lambda_0$ ,  $\lambda_0 \in \Lambda^1 S$ , be the connection form corresponding to the Sasakian structure, hence

$$d\lambda_0 = -2\omega.$$

Let  $\gamma : S \rightarrow S$  be an isometry, thus  $\gamma^*\omega = \omega$ . We want to construct  $\tilde{\gamma} : N \rightarrow N$ , such that

$$\tilde{\gamma}(t, z) = (\varphi(t, z), \gamma(z)),$$

and such that  $\tilde{\gamma}_*\partial_t = \partial_t$  and  $\tilde{\gamma}^*\lambda = \lambda$ . The first condition implies that  $\partial_t\varphi = 1$ , thus we can re-write

$$\tilde{\gamma}(t, z) = (t + f(z), \gamma(z)),$$

and therefore

$$\tilde{\gamma}^*\lambda = d(t + f(z)) + \tilde{\gamma}^*\lambda_0.$$

$\tilde{\gamma}$  is a Sasakian automorphism iff

$$df = \lambda_0 - \tilde{\gamma}^*\lambda_0. \tag{43}$$

$f$  can be determined, for the contractible surface  $S$ , iff the right hand side is a closed form. But

$$d\lambda_0 = \omega = \tilde{\gamma}^*\omega = \tilde{\gamma}^*d\lambda_0,$$

therefore  $d(\lambda_0 - \tilde{\gamma}^*\lambda_0) = 0$ . □

We study now the isometry group of the basis  $S$ , for the two cases:  $S = \gamma$  and  $S = \mathbf{H}$ . The following results are classical and their proofs elementary. The less obvious point is the case 2. from Lemma 4.12, and this follows also, for example, from [13].

**Lemma 4.11.** *Let  $\omega = f\omega_0$ , with  $f$  a positive function, and  $\omega_0 = dz_1 \wedge dz_2$ , a hermitian metric on  $\mathbb{C}$ . Then  $G = \text{Isom}(\omega)$  is a closed subgroup of  $\mathbb{C} \rtimes S^1 = \text{Isom}(\omega_0)$  and has the following form:*

- (1)  $\dim G = 0$ ; then  $G$  is a finite extension of a lattice  $L := \mathbb{Z}V \times \mathbb{Z}W$  by a subgroup  $G/L$  of the lattice automorphism group (which has 2, 4 or 6 elements);

- (2)  $\dim G = 1$ ; then  $G$  is an extension of  $L = \mathbb{Z}V \oplus \mathbb{R}W$ , with  $V, W$  linearly independent, by a subgroup  $G/L$  of  $\text{Aut}(L) = \{\pm I\}$ ;
- (3)  $G = \mathbb{C} \rtimes S^1$ .

**Lemma 4.12.** *Let  $\omega = f\omega_0$ , with  $f$  a positive function and  $\omega_0 = \frac{1}{z_2}dz_1 \wedge dz_2$ , a hermitian metric on  $\mathbf{H}$ . Then  $G = \text{Isom}(\omega)$  is a closed subgroup of  $PSL(2, \mathbb{R}) = \text{Isom}(\omega_0)$  and has the following form:*

- (1)  $\dim G = 0$ ; then  $G$  is a finite extension of a fuchsian lattice  $L := \pi_1(S_0)$  (here  $S_0 := \mathbf{H}/L$  is a Riemann surface of genus  $g > 1$ ) by a subgroup  $G/L$  of the lattice automorphism group (which is finite);
- (2)  $\dim G = 1$ ; then  $G$  is conjugated to

$$G_a := \left\{ \begin{pmatrix} a^k & a^k b \\ 0 & a^{-k} \end{pmatrix} \mid k \in \mathbb{Z}, b \in \mathbb{R} \right\},$$

where  $a > 1$  is fixed;

- (3)  $G = PSL(2, \mathbb{R})$ .

Now we come to the core of the argument. We want to show that we can modify the Sasakian structure on  $M_i$ , leaving it  $\Gamma_i$ -invariant, such that the new Sasakian structure is homogeneous.

**Lemma 4.13.** *Let  $N \rightarrow \S$  be a Sasakian manifold as above (with  $S \simeq \mathbb{C}$  or  $\mathbf{H}$ ). There exists a constant  $c > 0$  and a 1-form  $\alpha \in \Lambda^1 S$  such that*

- (1)  $\gamma^* \alpha = \alpha$ ,  $\forall \gamma \in \text{Isom}(S, \omega)$ ;
- (2)  $d\alpha = (c - f)\omega_0$ .

*Proof.* We use the description of the possible isometry groups as given in the Lemmas 4.11 and 4.12. In the first case of both lemmas,  $S/L$  is a compact Riemann surface, and all  $G$ -invariant data on  $S$  is equivalent to corresponding  $G/L$ -invariant data on  $S/L$ . The existence of an  $L$ -invariant  $\alpha$  as required is equivalent to

$$\int_{S/L} (c - f)\omega_0 = 0,$$

which determines  $c > 0$ . On the other hand, if  $\alpha \in \Lambda^1(S/L)$  satisfies

$$d\alpha = (c - f)\omega_0, \tag{44}$$

then  $\gamma^* \alpha$  also satisfies the same equation, for any  $\gamma \in G/L$ . Therefore,

$$\alpha_o := \left( \sum_{\gamma \in G/L} \gamma^* \alpha \right) / |G/L|,$$

where  $|G/L|$  is the number of elements of  $G/L$ , is a  $G/L$ -invariant 1-form satisfying (44), thus  $\alpha_o$  induces on  $S$  a 1-form with the required properties.

In the case 3. of both lemmas 4.11 and 4.12, the function  $f$  is constant, thus we trivially set  $c := f$  and  $\alpha := 0$ .

The remaining case is when  $\dim G = 1$ , and we will treat the case  $S = \mathbb{C}$  and  $S = \mathbf{H}$  distinctly.



Let  $S = \mathbb{C}$ . We choose a coordinate system on  $\mathbb{C}$  such that  $V \in \mathbb{R}$  and  $W = iw$ ,  $w > 0$  (with the notations of Lemma 4.11). The  $G$ -invariance of  $f$  implies that  $f$  depends only on  $z_2$  and is  $w$ -periodic.

We write  $\alpha := \alpha_1 dz_1 + \alpha_2 dz_2$  and, because of the required  $G$ -invariance of  $\alpha$ , we conclude that  $\alpha_1, \alpha_2$  should depend on  $z_2$  alone (and they should be  $w$ -periodic); as for the condition (44) the component  $\alpha_2 dz_2$  is irrelevant, we will set it to be zero.

In this setting, the claim is equivalent to: there exists a constant  $c > 0$  and a function  $\alpha_1 : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\begin{aligned} \alpha_2(z_2 + w) &= \alpha_2(z_2) \\ -\alpha_2' &= (c - f). \end{aligned} \quad (45)$$

A solution (hence all) of the second line satisfies the first line iff

$$c = \frac{1}{w} \int_0^w f(t) dt.$$

This proves the claim for  $S = \mathbb{C}$  and  $G = \mathbb{R} + iw\mathbb{Z}$ . To prove it for the extension of this group by  $\{\pm 1\}$ , we take the mean value between a solution  $\alpha_2$  of (45) and  $t \rightarrow \alpha_2(-t)$  (as we did for the case when  $\dim G = 0$ ).

Let now  $S = \mathbf{H}$ . After choosing some appropriate coordinates  $z_1 + iz_2$ ,  $z_1 \in \mathbb{R}$ ,  $z_2 > 0$ , on  $\mathbf{H}$ , we will suppose that the group  $G$  is equal to the group  $G_a$  from Lemma 4.12.  $f$ , being  $G$ -invariant, depends thus on  $z_2$  alone, moreover

$$f(a^2 z_2) = f(z_2). \quad (46)$$

At this moment, we make the change of variable

$$z_2 := e^y$$

and set  $a_0 := \ln a^2$ . We re-write (46):

$$f(y + a_0) = f(y), \quad \forall y \in \mathbb{R}. \quad (47)$$

As before, the coefficients of the 1-form  $\alpha$  must depend on  $z_2$  alone and, using the variable  $y$ , the required 1-form  $\alpha$  has to be  $\alpha = \beta(y)e^{-y}dz_1$ , with  $\beta$  a function, and the  $G$ -invariance of  $\alpha$  is equivalent to

$$\beta(y + a_0) = \beta(y), \quad \forall y \in \mathbb{R}. \quad (48)$$

The differential equation (44) is equivalent to

$$\beta' - \beta = c - f. \quad (49)$$

This equation is an affine differential equation that has global solutions on  $\mathbb{R}$ . If  $\beta_1, \beta_2$  are such solutions, their difference satisfies the associated linear equation, hence there is a constant  $p \in \mathbb{R}$  such that

$$\beta_1(y) - \beta_2(y) = pe^y.$$

Because  $(c - f)$  is  $a_0$ -periodic, for every solution  $\beta$  of (49),  $y \mapsto \beta(y + a_0)$  satisfies (49) as well, hence

$$\beta(y + a_0) - \beta(y) = pe^y,$$

for some  $p \in \mathbb{R}$ . Therefore,

$$\beta_0(y) := \beta(y) + \frac{p}{e^{a_0} - 1} e^y$$

satisfies (49) and is also  $a_0$ -periodic, as required. This proves the claim for  $S = \mathbf{H}$  and  $\dim G = 1$ , where we note that, unlike in the other cases,  $c$  can be arbitrarily chosen.  $\square$

The consequence of this Lemma is that the contact form  $\lambda_0 := \lambda + \alpha$ , together with the pre-Sasakian structure on  $N$  determines another  $G$ -invariant Sasakian structure on  $N$ , for which the automorphism group is maximal (4-dimensional; in particular  $(N, \xi, J, \lambda_0)$  is homogeneous).

As the space of hermitian metrics is convex, the passage from  $\omega$  to  $c\omega_0$  can be made smoothly, therefore the Sasakian structure on  $N$  is isotopic (through  $G$ -invariant Sasakian structures, and also, without changing the pre-Sasakian structure) to the Sasakian structure corresponding to a metric of constant curvature. This result is also true for  $N \simeq S^3$ , but there an additional deformation step is involved, that keeps the  $CR$  structure but not the pre-Sasakian structure.

The Sasakian structures corresponding to a principal bundle over a Riemann surface of constant curvature  $\kappa$  are left-invariant Sasakian structures on the Lie groups  $\widetilde{SL}(2, \mathbb{R})$  (for  $\kappa < 0$ ),  $Nil^3$  (for  $\kappa = 0$ ) and  $SU(2)$  (for  $\kappa > 0$ ) [4], [5], [6].

The Sasakian automorphisms of these homogeneous Sasakian manifolds are 4-dimensional, and are equal to

- (1)  $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}/\mathbb{Z}$  for  $\kappa < 0$ ; here, the factor  $\mathbb{R}$  is the Reeb flow, which at some times produces the central elements of  $\widetilde{SL}(2, \mathbb{R})$ , hence the quotient by  $\mathbb{Z}$ ;
- (2)  $Nil^3 \rtimes S^1$  for  $\kappa = 0$ ; here, the factor  $S^1$  comes from the rotations in  $\mathbb{C}$ , the space of the Reeb orbits;  $S^1$  acts on the Heisenberg group  $Nil^3$  by rotations on the contact plane;
- (3)  $SU(2) \times S^1/\{\pm 1\}$  for  $\kappa > 0$ , where  $S^1$  is the Reeb flow, and, as in the case  $\kappa < 0$ , the center  $\{\pm 1\}$  of  $SU(2)$  is common to it and the Reeb flow.

□

We have shown that the hermitian structure of a generic compact GCE manifold of dimension 6 (complex dimension 3) can be deformed to a locally homogeneous one. We also know that the universal covering  $\tilde{M}$  is a product of two of the three standard simply connected, complete, homogeneous Sasakian 3-manifolds (the Lie groups  $\widetilde{SL}(2, \mathbb{R})$ ,  $Nil^3$  and  $SU(2)$ ). The topological structure of the compact quotient  $M$  depends, thus, on the group  $\pi_1(M) = \Gamma$ . This group is not necessarily a product of lattices in the two factors, as the following example shows:

**Example.** Let  $\gamma_{1,2} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}$  be isomorphisms of  $\mathbb{C} \times \mathbb{C}$ :

$$\gamma_1(z, w) := (z + 1, qw), \quad \gamma_2(z, w) := (z + i, w + 1).$$

The projection of  $\gamma_{1,2}$  on the isometry groups of the factors is discrete and cocompact, provided that  $q$  is a root of unity of order 3, 4 or 6. If we consider some lifts

$$\tilde{\gamma}_{1,2} : Nil^3 \times Nil^3 \rightarrow Nil^3 \times Nil^3$$

of  $\gamma_{1,2}$  as  $\xi_{1,2}$ -preserving isometries of the Sasakian product  $\tilde{M} := Nil^3 \times Nil^3$  (which can be done as in Lemma 3.7), then they generate a discrete group of GCE isometries  $\Gamma$  on  $\tilde{M}$ .

The action of  $\gamma_{1,2}$  generates a cocompact group acting on  $\mathbb{C} \times \mathbb{C}$ , because

$$a(z, w) := \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}(z, w) = (z, w + q - 1)$$

and

$$b(z, w) := \gamma_1^2 \gamma_2 \gamma_1^{-2} \gamma_2^{-1}(z, w) = (z, w + q^2 - 1),$$

thus every point  $(z, w) \in \mathbb{C} \times \mathbb{C}$  can be brought, first by  $c_1$  and  $c_2$ , to a point  $(z_0, w')$ , with  $|z_0| < 2$ , then, using  $a$  and  $b$ , we get to a point  $(z_0, w_0)$ , with  $|w_0| < 2$  as well.

On the other hand, it can be shown that the group  $\Gamma$  contains some Reeb translations, such that the action of  $\Gamma$  on  $\tilde{M}$  is cocompact as well.

This latter claim follows from

**Lemma 4.14.** *The commutator of two translation lifts  $\tilde{\tau}_V, \tilde{\tau}_W$  in the Sasakian automorphism group of  $N \rightarrow \mathbb{C}$  is zero iff  $V, W$  are linearly dependent. More precisely, if the metric on  $\mathbb{C}$  is flat (hence  $N \simeq Nil^3$ ), the commutator of  $\tilde{\tau}_V, \tilde{\tau}_W$  is a Reeb translation by a shift that is proportional with the area of the parallelogram generated by  $V$  and  $W$ .*

*Proof.* Note that every vector field  $X$  on  $\mathbb{C}$  has a unique horizontal lift to  $N$ : it is the vector field  $\tilde{X} \in H$  (where  $H$  is the orthogonal space to the Reeb field) that projects on  $X \in T\mathbb{C}$ . The flow of a translation  $\tau_V$  can therefore be lifted to the flow  $\phi^V$  of a horizontal vector field  $\tilde{V} \in H_i$ . Note that this flow  $\phi^V$  does not preserve the contact structure on  $M_i$  (in fact  $\Lambda_{\tilde{V}}\lambda = -2\tilde{J}\tilde{V} \neq 0$ ), but it preserves the pre-Sasakian structure. However, if  $\tau_V \in \text{Isom}(\omega)$ , Lemma 3.7 implies that there exists a lift  $\tilde{\tau}_V$  acting on  $N$  by Sasakian automorphisms, i.e., in particular,  $\tilde{\tau}_V^*\lambda = \lambda$ . We can thus assume (after possibly composing  $\tilde{\tau}_V$  with some element in  $\Phi$ , the group of Reeb flows) that  $\tilde{\tau}_V(x_0) = \phi_1^{\tilde{V}}(x_0)$ , for some (hence for all)  $x_0 \in p_i^{-1}(0)$ . Here  $\phi_t^{\tilde{V}}$  is the flow of the vector field  $\tilde{V}$  at time  $t$ .

We can use the replacement  $\tilde{\tau}_V$ , resp.  $\tilde{\tau}_W$  (defined such that  $\tilde{\tau}_W(x_0) = \phi_1^{\tilde{W}}(x_0)$ ,  $\forall x_0 \in p_i^{-1}(V)$ ), because the composition with a central element in  $\Phi_i$  does not change the commutator of two elements.

It follows that

$$\tilde{\tau}_W \circ \tilde{\tau}_V(x_0) = \phi_1^{tW} \circ \phi_1^{\tilde{V}}(x_0),$$

and we want to determine  $(\tilde{\tau}_V)^{-1} \circ \tilde{\tau}_W \circ \tilde{\tau}_V(x_0)$ . Note that all elements of  $G_i$  that project on translations of  $\mathbb{C}$  (and thus commute with the constant vector fields  $V$  and  $W$ ) preserve the lifted vector fields  $\tilde{V}, \tilde{W}$ . Therefore,  $(\tilde{\tau}_V)^{-1}$  sends the integral curve of  $\tilde{W}$  containing  $\tilde{\tau}_V(x_0)$  to the integral curve of  $\tilde{W}$  through  $x_0$ . We conclude that

$$(\tilde{\tau}_V)^{-1} \circ \tilde{\tau}_W \circ \tilde{\tau}_V(x_0) = \phi_{-1}^{\tilde{W}}(x_0).$$

We suppose now that  $\tilde{\tau}_W \circ \tilde{\tau}_V = \tilde{\tau}_v \circ \tilde{\tau}_w$ . This implies that  $\tilde{\tau}_W \circ \tilde{\tau}_V(x_0)$  and  $(\tilde{\tau}_V)^{-1} \circ \tilde{\tau}_W \circ \tilde{\tau}_V(x_0)$  are connected by an integral curve of  $\tilde{V}$ , the image through  $(\tilde{\tau}_W)^{-1}$  of the integral curve of  $\tilde{V}$  through  $x_0$ .

This means that the parallelogram  $P$  in  $\mathbb{C}$  through  $0, V, W, V+W$  lifts to a closed horizontal curve  $C$  in  $M_i$ , thus there is a local section  $\sigma$  of  $M_i \rightarrow \mathbb{C}$  such that  $\sigma(\partial P) = C$ . However, by Stokes' Theorem, we have

$$\int_C \lambda = \int_{\partial P} \sigma^* \lambda = \int_P \sigma^* d\lambda.$$

The left hand side is zero because  $\lambda$  vanishes on the horizontal curve  $C$ , and the right hand side is the area of  $P$  for the metric induced on  $\mathbb{C}$  by the Sasakian metric on  $M_i$ , contradiction.

If the sequence of lifted segments does not close, we artificially close it by adding a piece of Reeb orbit. Then, using Stokes yields the desired result.  $\square$

We see that the groups acting freely, properly discontinuously and with compact quotient on  $G_1 \times G_2$ , for  $G_i \in \{\widetilde{SL}(2, \mathbb{R}), Nil^3, SU(2)\}$  do not necessarily preserve the leaves of the two orthogonal foliations. Their general structure deserves further research.

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